A New Class of Integer Solutions to Homogeneous Cubic Equation with Four Unknowns $x^3 + y^3 = 42zw^2$

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D.O.I: 10.56201/ijasmt.v10.no2.2024.pg1.7

Abstract

This paper aims at presenting different choices of non-zero solutions in integers to the homogeneous cubic equation with four unknowns given by $x^3 + y^3 = 42zw^2$. Various sets of integer solutions are obtained by Scrutiny, substitution technique and method of factorization.

Keywords *: Homogeneous cubic equation ,Cubic equation with four unknowns, Integer solutions*

Introduction

Diophantine equations , one of the areas of number theory ,occupy a pivotal role in the realm of mathematics and have a wealth of historical significance. It is well-known that Diophantine equations are rich in variety. Particularly , finding integer solutions to homogeneous cubic equation with four unknowns is a topic of current research. While collecting problems on the same , the article presented in [1] was noticed and the authors have obtained a few sets of integer solutions .However ,there are many more fascinating patterns of solutions in integers . The main thrust of this paper is to exhibit other solution patterns to the homogeneous cubic equation with four unknowns $x^3 + y^3 = 42zw^2$.

Technical procedure

Consider the homogeneous cubic equation with four unknowns given by

$$x^3 + y^3 = 42zw^2 \tag{1}$$

At the outset ,by scrutiny , it is seen that (1) is satisfied by the following integer quadruples

$$(x, y, z, w) = (10,8,9,2), (-1,-5,-3,1), (8,-2,3,2), (16,-4,6,4),$$
$$(42^s m(m^3 + n^3), 42^s n(m^3 + n^3), 42^{s-1}(m^3 + n^3)^2, 42^s (m^3 + n^3))$$

However , there are some more choices of integer solutions to (1) that are illustrated below:

The substitution of the linear transformations

$$x = u + v, y = u - v, z = u, u \neq v \neq 0 \tag{2}$$

in (1) leads to the ternary quadratic equation

$$u^2 + 3v^2 = 21w^2 \tag{3}$$

Illustration 1

Let

$$w = 4(a^2 + 3b^2) \tag{4}$$

The integer 21 on the R.H.S. of (3) is written as

$$21 = \frac{(9 + i\sqrt{3})(9 - i\sqrt{3})}{4} \tag{5}$$

Using (4) & (5) in (3) and employing the method of factorization ,consider

$$u + i\sqrt{3}v = 2(9 + i\sqrt{3})(a + i\sqrt{3}b)^2$$

On equating the coefficients of corresponding terms ,we have

$$u = 18(a^2 - 3b^2) - 12ab \,,$$
$$v = 2\,(a^2 - 3b^2) + 36ab \,.$$

In view of (2) , one has

$$x = 20\,(a^2 - 3b^2) + 24ab \,,$$
$$y = 16\,(a^2 - 3b^2) - 48\,ab \,, \qquad (6)$$
$$z = 18\,(a^2 - 3b^2) - 12\,ab \,.$$

Thus , (4) and (6) satisfy (1) .

Illustration 2

Write (3) as

$$u^2 + 3v^2 = 21w^2 * 1 \qquad (7)$$

Let the integer 21 on the R.H.S. of (7) be written as

$$21 = (3 + i2\sqrt{3})\,(3 - i2\sqrt{3}) \qquad (8)$$

Consider the integer 1 0n the R.H.S. of (7) as

$$1 = \frac{(1 + i4\sqrt{3})\,(1 + i4\sqrt{3})}{49} \qquad (9)$$

Assume

$$w = (a^2 + 3b^2) \qquad (10)$$

Using (8) ,(9)& (10) in (7) and employing the method of factorization ,

consider

$$u + i\sqrt{3}\,v = \frac{(1 + i4\sqrt{3})}{7}\,(3 + i2\sqrt{3})\,\,(a + i\sqrt{3}\,b)^2$$

On equating the coefficients of corresponding terms ,we have

$$u = -3(a^2 - 3b^2) - 12ab \,,$$
$$v = 2\,(a^2 - 3b^2) - 6ab \,.$$

In view of (2) , one has

$$x = -(a^2 - 3b^2) - 18ab\,,$$
$$y = -5(a^2 - 3b^2) - 6ab\,, \qquad (11)$$
$$z = -3(a^2 - 3b^2) - 12ab\,.$$

Thus , (10) and (11) satisfy (1) .

Note 1

It is to be seen that ,apart from (9) , we have

$$1 = \frac{(3r^2 - s^2 + i\,2rs\sqrt{3})\,(3r^2 - s^2 - i\,2rs\sqrt{3})}{(3r^2 + s^2)^2}$$

Repeating the above process ,a different set of integer solutions to (1)

is obtained.

Illustration 3

Rewrite (3) as

$$21w^2 - u^2 = 3v^2 \qquad (12)$$

Assume

$$v = 21a^2 - b^2 \qquad (13)$$

Express the integer 3 on the R.H.S. of (12) as

$$3 = \frac{(\sqrt{21} + 3)\,(\sqrt{21} - 3)}{4} \qquad (14)$$

Using (13) & (14) in (12) and employing the method of factorization ,

consider

$$\sqrt{21}\,w + u = \frac{(\sqrt{21} + 3)}{2}\,(\sqrt{21}\,a + b)^2$$

On equating the coefficients of corresponding terms ,we have

$$u = \frac{3(21a^2 + b^2) + 42ab}{2}\,,$$
$$w = \frac{(21a^2 + b^2) + 6ab}{2}$$

As our aim is to obtain integer solutions ,replacing a by $2A$ and b by $2B$,

we have

$$u = 6(21A^2 + B^2) + 84AB\,,$$
$$v = 4(21A^2 - B^2)\,,$$

and

$$w = 2(21A^2 + B^2) + 12AB \qquad (15)$$

In view of (2) , we have

$$x = 210A^2 + 2B^2 + 84AB\,,$$
$$y = 42A^2 + 10B^2 + 84AB\,, \qquad (16)$$
$$z = 6(21A^2 + B^2) + 84AB\,.$$

Thus , (15) and (16) satisfy (1) .

Note 2

It is to be seen that ,apart from (14) , we have

$$3 = (2\sqrt{21} + 9)\,(2\sqrt{21} - 9)$$

Repeating the above process ,a different set of integer solutions to (1)

is obtained.

Illustration 4

Rewrite (3) as

$$21w^2 - 3v^2 = u^2 * 1 \qquad (17)$$

Assume

$$u = 21a^2 - 3b^2 \qquad (18)$$

Express the integer 1 on the R.H.S. of (17) as

$$1 = \frac{(\sqrt{21} + 2\sqrt{3})\,(\sqrt{21} - 2\sqrt{3})}{9} \qquad (19)$$

Using (18) & (19) in (17) and employing the method of factorization , consider

$$\sqrt{21}\,w+\sqrt{3}\,v=\frac{(\sqrt{21}+2\sqrt{3})}{3}\;(\sqrt{21}\,a+\sqrt{3}\,b)^2$$

On equating the coefficients of corresponding terms ,we have

$$v=14a^2+2b^2+14ab\,,$$

and

$$w=7a^2+b^2+4ab\,.$$

In view of (2) , we have

$$x=35a^2-b^2+14ab,$$
$$y=7a^2-5b^2-14ab,$$
$$z=.21a^2-3b^2$$

Thus , the above values of x, y, z, w satisfy (1).

Note 3

It is to be seen that ,apart from (19) , we have

$$1=\frac{(2\sqrt{21}+5\sqrt{3})\,(2\sqrt{21}-5\sqrt{3})}{9}$$

Repeating the above process ,a different set of integer solutions to (1)

is obtained.

Illustration 5

Express (3) in the form of ratio as

$$\frac{u+3w}{2w+v}=\frac{3(2w-v)}{u-3w}=\frac{P}{Q}\,,Q\neq 0 \tag{20}$$

Solving the above system of double equations , one has

$$u=3P^2+12PQ-9Q^2\,,$$
$$v=-2P^2+6PQ+6Q^2\,, \tag{21}$$

and

$$w = 3Q^2 + P^2 \tag{22}$$

From (21) and (2) ,we have

$$\begin{aligned} x &= P^2 + 18PQ - 3Q^2 \ , \\ y &= 5P^2 + 6PQ - 15Q^2 \ , \\ z &= 3P^2 + 12PQ - 9Q^2 \, . \end{aligned} \tag{23}$$

Thus , (1) is satisfied by (22) and (23).

Conclusion

In this paper , we have illustrated various ways of obtaining non-zero distinct integer solutions to the homogeneous cubic equation with four unknowns given by $x^3 + y^3 = 42zw^2$ and these solutions are different from the solutions presented in [1]. To conclude , one may attempt for getting integer solutions to other choices of homogeneous cubic equations with four unknowns .

References

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